Non-local length estimators and concave functions

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Abstract

In a previous work [1], the authors introduced the *Non-Local Estimators* (NLE), a wide class of polygonal length estimators including the sparse estimators and a part of the DSS ones. NLE are studied here under concavity assumption and it is shown that concavity almost doubles the multigrid converge rate w.r.t. the general case. Moreover, an example is given that proves that the obtained convergence rate is optimal. Besides, the notion of *biconcavity* relative to a NLE is proposed to describe the case where the digital polygon is also concave. Thanks to a counterexample, it is shown that concavity does not imply biconcavity. Then, an improved error bound is computed under the biconcavity assumption.

Keywords: digital geometry, length estimation, multigrid convergence

1. Introduction

This article is the second of a pair devoted to the study of the multigrid convergence of length estimators. For short, the considered length estimators are based on a polygonal approximation of the digitized function whose edge discrete sizes tend in mean toward infinity, as the grid step tends toward zero. Indeed, it is known that length estimators using fixed size edges, even with suitable weights, do not converge in the general case and it is likely that this result could be extend to estimators using edges of bounded sizes, weighted or not. In the first article [1], we introduced the notion of *non local estimator* (NLE), a polygonal estimator using edges whose mean discrete size tend toward infinity and, among the NLE, we considered in particular the *M-sparse estimators* (MSE) whose *true* edge lengths (taking into account the grid step) tend toward zero in mean. We proved that a MSE, or a NLE *close* to a MSE, has the multigrid convergence property. In the present article, we focus on the improvement brought by the concavity assumption on the multigrid convergence speed for the NLE. Indeed, we know from a previous work [2], that convexity doubles the convergence rate of the *sparse estimators* the most regular MSE. This is not exactly the case in the more general setting of the NLE but nevertheless

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we show that the convergence is significantly accelerated by the concavity for a wide class of continuous functions that satisfy a Lipschitz condition on the left and the right derivative. Moreover, we introduce the notion of *biconcavity* which expresses that both the continuous curve and the polygonal line used for the length estimation are concave. This notion was implicitly used in [3, theorem 13] to prove the multi-grid convergence of the *maximal digital straight segment estimator* (MDSSE). Under the biconcavity assumption, we establish a result that fit our observations on the convergence speed of the MDSSE for the natural logarithm function.

The paper is organized as follows. In Section 2, some necessary notations and conventions are recalled, as are the NLE convergence properties in the general case. Two theorems on the multigrid convergence rate of NLE and MSE for concave continuous functions are given in Section 3. An experiment exemplifies the results. Section 4 is devoted to the biconcavity. A sufficient condition for this property is presented and we state our third theorem on the convergence rate. Section 5 concludes the article. The reader will also find in Appendix A an example of a concave function for which our best upper bound for the convergence rate is reached, indicating that this bound cannot be improved in the general case. Moreover an example of a concave function whose digitization family has convex pairs of arbitrary long consecutive chords for an infinity of grid steps is exhibited. Eventually, Appendix B gathers the technical lemmas used in Sections 3 and 4.

2. Background and previous results

In this section, we give our notations and we recall the notion of Non-Local Estimators (NLE) introduced in [1].

2.1. Digitization models

This paper is focused on the digitization of function graphs. So, let us consider a continuous function $g : [a, b] \to \mathbb{R}$ ($a < b$), its graph $\mathcal{C}(g) = \{(x, g(x)) \mid x \in [a, b]\}$ and a positive real number r, the *resolution*. We assume to have an orthogonal grid in the Euclidean space $\mathbb{R}^2$ whose set of grid points is $h\mathbb{Z}^2$ where $h = 1/r$ is the *grid spacing*. We use the following notations: $\lfloor \cdot \rfloor$ is the floor function and $\lceil \cdot \rceil$ is the ceil function. For $i \leq j$ two integers, $[\![i, j]\!]$ stands for $[i, j] \cap \mathbb{Z}$. The *h-digitization* of the function g is the discrete function $\mathcal{D}(g, h)\colon [\![\lceil a/h \rceil, \lfloor b/h \rfloor]\!] \to \mathbb{Z}$ defined by $\mathcal{D}(g, h)(k) = \lfloor g(kh)/h \rfloor$. Provided the slope of g is limited by 1 in modulus, the graph of $\mathcal{D}(g, h)$ is an 8-connected digital curve. Nevertheless, in this article, we make no assumption on the slope of the function g.

2.2. Non-local length estimators (NLE)

For any continuous function $f\colon [a, b] \to \mathbb{R}$, $L(f)$ denotes the length of the graph $\mathcal{C}(f)$ according to Jordan's definition of length:

$$L(f) = \sup_{a = x_0 < x_1 < \cdots < x_n = b} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$

where the supremum is taken over all the possible partitions of [a, b] and n is unbounded. The reader can find in [1] a description of the classical length estimators.

Let us now recall the key notions in the definition of the NLEs.

- A *pattern function* is a function that maps a discrete curve Γ and a grid spacing h to a partition of the domain of Γ.

  Let $\mathcal{A}$ and $\mathcal{B}$ be two pattern functions. We say that $\mathcal{A}$ is *finer* than $\mathcal{B}$, we write $\mathcal{A} \prec \mathcal{B}$, if for any discrete curve Γ and any grid step h, the partition $\mathcal{A}(G, h)$ is finer than the partition $\mathcal{B}(G, h)$.

- Let $\alpha \in \overline{\mathbb{R}} = [-\infty, +\infty]$ be any non-zero real number. When σ is a partition of some interval $I \subset \mathbb{R}$, the α-th power mean of the σ subinterval length sequence $(x_i)_{i=0}^n$ is defined for $\alpha \in \mathbb{R}$ by

  $$M_\alpha((x_i)_{i=0}^n) = \left(\frac{1}{n}\sum_{i=0}^{n} {x_i}^{\alpha}\right)^{\frac{1}{\alpha}},$$

  and $M_{+\infty}((x_i)_{i=0}^n) = \max((x_i)_{i=0}^n)$, $M_{-\infty}((x_i)_{i=0}^n) = \min((x_i)_{i=0}^n)$ in the other cases.

  An *α-pattern function $\mathcal{A}$ on* a set of rectifiable functions C is a pattern function such that, for any function $g \in C$, $\lim_{h\to 0} M_\alpha(\mathcal{A}(\mathcal{D}(g, h), h)) = +\infty$.

- An *(α, β)-pattern function* ($\beta \in \overline{\mathbb{R}}$) $\mathcal{A}$ *on C* is an α-pattern function such that, for any function $g \in C$, $\lim_{h\to 0} M_\beta(\mathcal{A}(\mathcal{D}(g, h), h)) \times h = 0$.

- An α-pattern function, resp. (α, β)-pattern function, is an α-pattern function, resp. (α, β)-pattern function, on the set of all rectifiable functions.

The *non-local length estimator* associated to an α-pattern function $\mathcal{A}$ maps a pair (G, h), consisting of a discrete curve and a grid step, to the length $L^{\mathrm{NL}}(\mathcal{A}, G, h)$ of an h-homothetic copy of the polyline whose vertices are the points of G with abscissas in $\mathcal{A}(G, h)$. Given a rectifiable function g, by abuse of notation, we write $L^{\mathrm{NL}}(\mathcal{A}, g, h)$ instead of $L^{\mathrm{NL}}(\mathcal{A}, \mathcal{D}(g, h), h)$ and also $\mathcal{A}(g, h)$ instead of $\mathcal{A}(\mathcal{D}(g, h), h)$. Let $H : (0, +\infty) \to \mathbb{N}^\star$. A *sparse estimator with step H* is a non-local length estimator whose pattern function $\mathcal{A}$ only depends on the grid step h and such that the partition $\mathcal{A}(G, h)$ has a constant step $H(h)$ but its last step which is not greater than $H(h)$.

The main result without concavity hypothesis is that NLE are convergent for Lipschitz functions. We recall below (Theorem 1) a result, proved in [1], that gives a bound on the error at the grid spacing h for Lipschitz functions whose derivatives are k-Lipschitz on any interval included in their domains ($k > 0$). Before stating Th. 1, we need first to complete the introduction to our notations.

Notations. We present some notations used throughout the remainder of the article. The first ones concern Euclidean objects. Thereby, they do not depend upon the grid spacing. The others are related to the grid spacing h and should be indexed by h. Nevertheless, as we never have to work with two different grid spacings, the h index is omitted to lighten the notations.

$I = [a, b]$ is an interval of $\mathbb{R}$ with a non-empty interior and $g\colon I \to \mathbb{R}$ is a Lipschitz function whose derivative is denoted g' (since g is Lipschitz-continuous, it is absolutely continuous and thus, g is differentiable almost everywhere [4, p. 145-148]). The function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is defined by $\varphi(x) = \sqrt{1 + x^2}$. Thus, one has $L(g) = \int_{[a,b]} \varphi \circ g'(t)\,\mathrm{d}t$.

Given some grid spacing $h > 0$, A, resp. B, is the smallest, resp. largest, integer such that $Ah \in I$, resp. $Bh \in I$. The functions g_{l}, g_{c}, g_{r} are resp. the restrictions of the function g to the intervals $[a, Ah]$, $[Ah, Bh]$, $[Bh, b]$. For any patern function $\mathcal{A}$, we write $M_\alpha^{\mathcal{A}}$, instead of $M_\alpha(\mathcal{A}(g, h))$ when there is no ambiguity. The number of subintervals in the partition $\mathcal{A}(g, h)$ is denoted $N^{\mathcal{A}}$, or just N when possible and the integers defining the partition $\mathcal{A}(g, h)$ are $A = a_0 < a_1 < \cdots < a_N = B$ ($A = b_0 < b_1 < \cdots < b_N = B$ for the partition $\mathcal{B}(g, h)$). In particular, for a sparse estimator with step H and a real α, the mean $M_\alpha(\mathcal{A}(G, h))$ lies between $H(h)$ and $H(h)(1 - 1/N)^{1/\alpha}$. Finally, two piecewise affine functions, $g_{\mathrm{c}}^{\mathcal{A}}$ and $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$, are defined. They interpolate the continuous function g_{c} and its digitization (actually, the h-homothetic copy of the digital curve $\mathcal{D}(g, h)$) according to the pattern function $\mathcal{A}$. The graph of $g_{\mathrm{c}}^{\mathcal{A}}$, resp $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$, is the polyline linking the points $\left(a_i h, g(a_i h)\right)_{i=0}^{N}$ which are in $\mathcal{C}(g)$, resp. the grid points $\left(a_i h, \lfloor \frac{g(a_i h)}{h} \rfloor h\right)_{i=0}^{N}$ which are in $h\mathbb{Z}^2$.

We are now able to state Th. 1 (see [1]).

Theorem 1. *Let $g\colon [a, b] \to \mathbb{R}$ be a k_1-Lipschitz function and $\mathcal{A}$ be a 1-pattern function. If there exist a $(1, \beta)$-pattern function $\mathcal{B}$, $\beta \in [1, +\infty]$, and a real ω such that, for any grid spacing h, $\| \lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor - \lfloor g_{\mathrm{c}}^{\mathcal{B}} \rfloor \|_\infty \leq \omega h$, then*

- *if $\beta = +\infty$, the non-local estimation $L^{\mathrm{NL}}(g, \mathcal{A}, h)$ converges toward the length of the curve $\mathcal{C}(g)$ as h tends to 0;*
- *if g' is k_2-Lipschitz on each interval included in its domain, we have*

$$L(g) - L^{\mathrm{NL}}(\mathcal{A}, g, h) \leq$$
$$Sh + ThM_1^{\mathcal{B}}(1 + (C^{\mathcal{B}})^2) + U\mathcal{H}^{\mathcal{B}} + V\left(\frac{1}{M_1^{\mathcal{A}}} + \frac{1}{M_1^{\mathcal{B}}}\right), \quad (1)$$

  where $S = 2\varphi(k_1)$, $T = k_2(b - a)/2$, $U = \varphi(k_1) - 1$, $V = (1 + 2\omega)\varphi'\Big(k_1 + 1/M_{-1}^{\mathcal{A}}\Big)(b - a)$ and $\mathcal{H}^{\mathcal{B}}$ is the measure of the union of the $B(g, h)$ subintervals on which g is not differentiable.

Furthermore, if $\mathcal{B}(g, h) \subseteq \mathcal{A}(g, h)$, the term $1/M_1^{\mathcal{A}} + 1/{M_1}^{\mathcal{B}}$ in the right hand side of Equation (1) can be replaced by $1/M_1^{\mathcal{B}}$.

Apart from the first one, the upper bounds that appear in the right hand side of Equation (1) can be improved in the case of concave functions.

3. Concave functions length estimation

In this section, we assume that the function g is concave on $[a, b]$. This implies in particular that g admits left and right derivatives, noted $\mathrm{d}_\ell g$ and $\mathrm{d}_r g$, at any point of (a, b) and is Lipschitz continuous on any closed subinterval of (a, b). We assume moreover that the one-sided derivatives of g are defined and Lipschitz[1] on $[a, b]$. In particular, g is Lipschitz on $[a, b]$. Under this new hypothesis, we can improve the bound on the convergence speed of the estimated length toward the true length of the curve $\mathcal{C}(g)$.

3.1. General case

Let $\mathcal{A}$ be a pattern function. The functions g_l, g_r, $g_\mathrm{c}^{\mathcal{A}}$ and $\lfloor g_\mathrm{c}^{\mathcal{A}} \rfloor$ are those defined in Paragraph *Notations* of Section 2.2. Firstly, we recall a bound on the errors due to the loss of the true left and right extremities of the curve $\mathcal{C}(g)$. Its proof can be found in [1].

Proposition 2 (Curve extremity error). *For any k-Lipschitz function g, we have*

$$L(g_\mathrm{l}) + L(g_\mathrm{r}) \leq 2\varphi(k)h.$$

Propositions 3 and 4 are improvements of Propositions 3 and 4 of [1] for concave curves. The first one gives an upper bound on the *discretization* error.

Proposition 3 (Error between curve and curve chords). *Let g be a concave function whose one-sided derivatives are defined and k-Lipschitz on $[a, b]$ $(k > 0)$. Then*

$$L(g_\mathrm{c}) - L(g_\mathrm{c}^{\mathcal{A}}) \leq \sum_{i=1}^{N} \frac{k^2}{4}(a_i - a_{i-1})^3 h^3 \leq \frac{k^2(b-a){M_3}^3}{4M_1}\, h^2. \qquad (2)$$

Proof. Note that the proof appeals to a technical result, Lemma 12, which is stated, and proved, in Appendix B.

Let us consider the partition $\sigma = h \cdot \mathcal{A}(g, h)$ of the interval $[Ah, Bh]$ and the piecewise affine function $g_\mathrm{c}^{\mathcal{A}+} \colon [Ah, Bh] \to \mathbb{R}$ defined by

$$g_\mathrm{c}^{\mathcal{A}+}(x) = \min\bigl(g(x_{i-1}) + \mathrm{d}_r g(x_{i-1})(x - x_{i-1}),\ g(x_i) - \mathrm{d}_\ell g(x_i)(x_i - x)\bigr)\ ,$$

where $[x_{i-1}, x_i]$ is the subinterval of the partition σ that contains x. Note that $g_\mathrm{c}^{\mathcal{A}+}(x_i)$, $0 \leq i \leq N$, is uniquely defined and is equal to $g(x_i)$.

Since g is concave, we have on the one hand $\mathrm{d}_r g(x_{i-1}) \leq g' \leq \mathrm{d}_\ell g(x_i)$ on any subinterval $[x_{i-1}, x_i]$ of σ and, on the other hand, $g_\mathrm{c}^{\mathcal{A}} \leq g_\mathrm{c} \leq g_\mathrm{c}^{\mathcal{A}+}$ on $[Ah, Bh]$. Therefore, we can apply Lemma 11 and Lemma 12 on each subinterval of the partition σ. Together with the hypothesis on the derivatives of g, this leads to the following inequalities.

[1] Since g is concave on $[a, b]$, it is equivalent to assume that $\mathrm{d}_\ell g$ — or $\mathrm{d}_r g$ — is k-Lipschitz for some $k > 0$, or that $\mathrm{d}_r g(x) - \mathrm{d}_\ell g(y) \leq k(y - x)$ for any x, y such that $a \leq x < y \leq b$.

$$L(g_c) - L(g_c^{\mathcal{A}}) \le L(g_c^{\mathcal{A}+}) - L(g_c^{\mathcal{A}}) \le \sum_{i=1}^{N}(x_i - x_{i-1})\,\frac{(\mathrm{d}_r g(x_{i-1}) - \mathrm{d}_\ell g(x_i))^2}{4}$$
$$\le \sum_{i=1}^{N} \frac{k^2}{4}(x_i - x_{i-1})^3 \le \frac{k^2 h^3 N}{4} {M_3}^3 \le \frac{k^2 h^2 (b-a)}{4}\frac{{M_3}^3}{M_1} \ .$$

Hence, the result holds. □

Inequality (2) has to be compared to the following one obtained in [1, Proposition 3] for a function g differentiable with a derivative k Lipschitz continuous:

$$L(g_c) - L(g_c^{\mathcal{A}}) \le \frac{k(b-a)}{2} h M_2 \ .$$

When the partition $\mathcal{A}(g, h)$ is roughly even, ${M_3}^3/M_1 \approx {M_2}^2$ and the upper bound is squared under the concavity assumption. In the worst case, we also note that

$$\frac{{M_3}^3}{M_1} = \frac{\sum (a_{i+1} - a_i)^3}{\sum (a_{i+1} - a_i)} \le \frac{\sum (a_{i+1} - a_i){M_{+\infty}}^2}{\sum (a_{i+1} - a_i)} \le (M_{+\infty})^2 \ . \quad (3)$$

Example 1. *The result given by Proposition 3 is illustrated on Fig. 1 with the natural logarithm on the interval* $[1, 2]$*, the sparse estimators with steps* $H(h) = h^{-\gamma}$ *where* $\gamma \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$ *and the MDSS estimator. The grid steps used for the plot are* $h = (2/3)^n$*,* $n \in [1, 40]$*. Then, for any* γ*,* $M_\alpha \approx h^{-\gamma}$ *(precisely,* $h^{-\gamma}(1-h)^\alpha \le M_\alpha \le h^{-\gamma}$*) and Eq.* (2) *gives the following expression for the discretization error*

$$L(g_c) - L(g_c^{\mathcal{A}}) = \tfrac{1}{4} h^{2(1-\gamma)} \ .$$

In Figure 1, the continuous lines stand for the error computed from the formula above, where the constant has been estimated from the data. We see that Eq. (2) *gives the right convergence rate though the given constant (*$1/4$*) is bigger than the empirical ones (between* 0.1 *and* 0.001*). This was expected mainly because Eq.* (2) *involves an upper bound for the second derivative while this derivative is not constant. Regarding the MDSS estimator, we just know from [5] that*

$$\Omega(h^{-1/3}) \le M_1 \le \mathcal{O}(h^{-1/3}\log(h^{-1})) \ .$$

So, we plotted two lines $\propto h^{4/3}$ *and* $\propto h^{4/3}\log^2(h^{-1})$ *that fit the data well.*

The following proposition gives an upper bound on the *quantization error*. It appeals to two pattern functions. Indeed, the pattern functions have been introduced in [1] to report on the behavior of two families of length estimators:

- sparse estimators [2] that use domain partitions $\mathcal{A}(G, h)$ that only depends upon the parameter h,

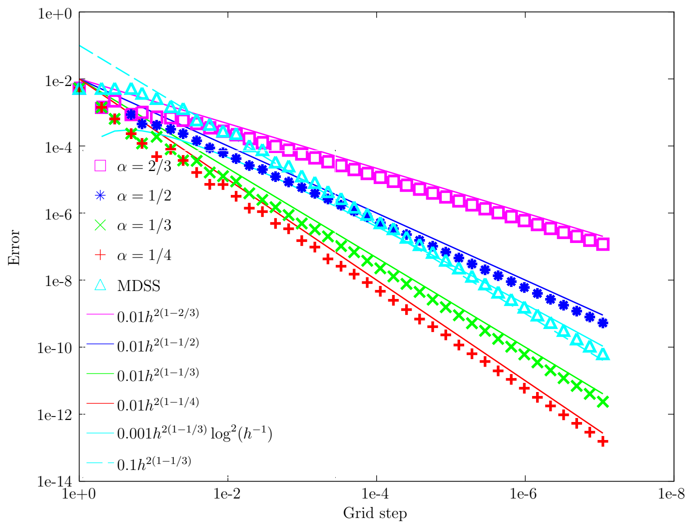

Figure 1: $\left|L(g_c) - L(g_c^{\mathcal{A}})\right|$ (see text).

- MDSS (Maximum Digital Straight Segments) that use domain partitions that only depend upon the discrete function G

(local estimators domain partitions depend neither upon h nor upon G and fail to converge). Since MDSS domain partitions depend on the function graph, one cannot assert anything about the 'true length' of the subsegments of a MDSS so the underlying pattern function of a MDSS is not in general an (α, β)-pattern function. Nevertheless, since by definition a MDSS is close to the curve, the resulting digital curve segmentation is not far from the segmentation produced by some (α, β)-pattern function. This is the reason why in the next proposition and in the proof of Theorem 6, we appeal to two pattern functions that are close to each other.

Proposition 4 (Error between curve chords and grid chords). *Let g be a concave function and $\mathcal{A}$ and $\mathcal{B}$ be two pattern functions such that $\mathcal{B} \prec \mathcal{A}$ and $g_c^{\mathcal{B}} - \lfloor g_c^{\mathcal{A}} \rfloor \le \omega h$ for some $\omega > 0$. Then*

$$\left|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| \le U \sum_{i=1}^{N^{\mathcal{B}}} \frac{h}{b_i - b_{i-1}} + V h \le U \frac{b-a}{M_{-1}^{B} M_{1}^{B}} + V h \ , \qquad (4)$$

where $U = \omega^2$ and $V = \max(g'(a), g'(a) - 2g'(b))$.

Proof. From the hypotheses, we have

$$\lfloor g_c^{\mathcal{A}} \rfloor \le g_c^{\mathcal{B}} \le \lfloor g_c^{\mathcal{A}} \rfloor + \omega h \ .$$

Let s_1 and s_2 be the slopes of the first and last segments of $g_c^{\mathcal{B}}$. Since g is concave, $g'(a) \geq s_1 \geq s_2 \geq g'(b)$. From Lemma 14, applied with $f_1 = \lfloor g_c^{\mathcal{A}} \rfloor$, $f_2 = g_c^{\mathcal{B}}$, $\sigma = h\mathcal{B}(g,h)$, $p = N^{\mathcal{B}}$ and $e = \omega h$, we derive

$$\begin{aligned} \left|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| &\leq U \sum_{i=1}^{N^{\mathcal{B}}} \frac{h}{b_i - b_{i-1}} + Vh \quad \text{for } \max(s_1, s_1 - 2s_2) \leq V \\ &\leq U \frac{N^{\mathcal{B}} h}{M_{-1}^{\mathcal{B}}} + Vh \leq U \frac{b-a}{M_{-1}^{\mathcal{B}} M_1^{\mathcal{B}}} + Vh \ . \end{aligned}$$

□

Example 2. *The result given by Proposition 4 is illustrated on Fig. 2 with the same function and patterns as in Example 1, taking each time $\mathcal{A} = \mathcal{B}$ (and $\omega = 1$). With the sparse estimators, we have, for any γ and α, $M_\alpha = \Theta(h^{-\gamma})$. For the MDSS estimator, we assume that, for any α, M_α is in $\Theta(h^{-1/3})$ or in $\Theta(h^{-1/3}\log(h^{-1}))$. Then, Eq.* (4) *gives the following upper bounds for the error $L(g_c^{\mathcal{A}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)$:*

- $\mathcal{O}\big(h^{\min(1,2\gamma)}\big)$ *for the sparse estimators;*
- $\mathcal{O}\big(h^{2/3}\big)$*, or* $\mathcal{O}\big(h^{2/3}/\log^2(h^{-1})\big)$*, for the MDSS estimator.*

The continuous lines in Fig. 2 correspond to these upper bounds. Though the behavior of the quantization error is less regular than the behavior of the discretization error, the observed convergence rates for the quantization errors fit again our upper bounds. Also, note that the observed constants, hidden in the big O, are smaller than the ones calculated from Eq. (4) *(from a factor of about 10).*

From Propositions 2, 3 and 4, we derive the following theorems on the convergence speed when the function g is concave. Compared to Theorem 1, concavity almost squares the convergence speed. In particular, the optimal step-size for uniform size algorithms remains unchanged ($H_\gamma(h) = \Theta(h^{-\frac{1}{2}})$) but the speed is improved up to h.

Theorem 5. *Let $\mathcal{A}$ be a $(-1,+\infty)$-pattern function. Let $g\colon [a,b] \to \mathbb{R}$ be a concave function whose one-sided derivatives are defined and Lipschitz on $[a,b]$. Then $L^{\mathrm{NL}}(\mathcal{A},g,h)$ converges toward $L(g)$ as h tends to zero and*

$$L(g) - L^{\mathrm{NL}}(\mathcal{A},g,h) = \mathcal{O}\left(\frac{h^2\left(M_3\right)^3}{M_1}\right) + \mathcal{O}\left(\frac{1}{M_{-1}M_1}\right) \ . \tag{5}$$

Proof. The function g satisfies the hypothesis of Propositions 2, 3 and 4. So

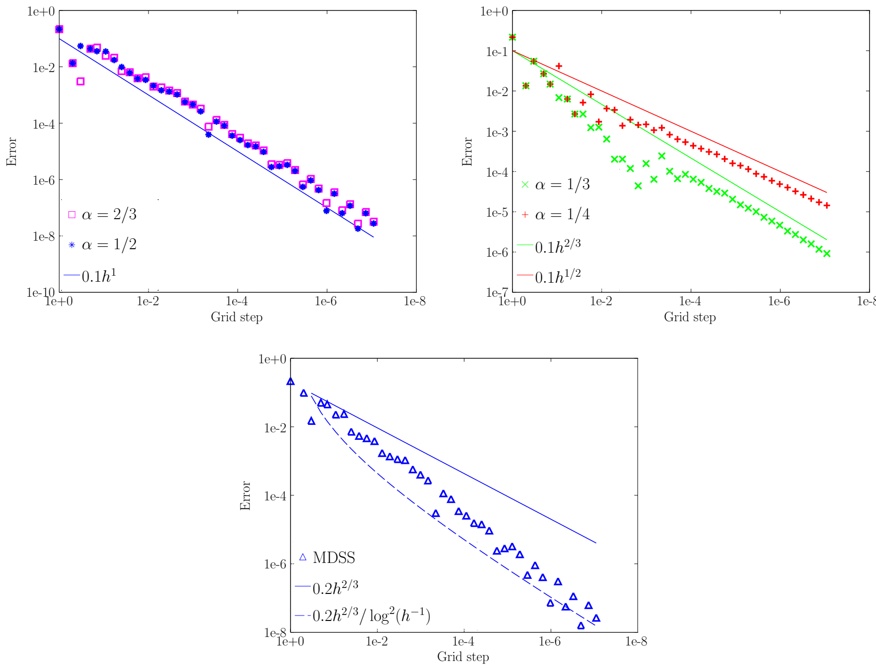

Figure 2: $\left|L(g_c^{\mathcal{A}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right|$ (see text).

we have

$$\left|L(g) - L(g_c)\right| = \mathcal{O}(h) \ ,$$

$$\left|L(g_c) - L(g_c^{\mathcal{B}})\right| = \mathcal{O}\left(\frac{h^2 \left(M_3\right)^3}{M_1}\right) \ ,$$

$$\left|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| = \mathcal{O}\left(\frac{1}{M_{-1}M_1}\right) + \mathcal{O}(h) \ .$$

Since $\alpha \mapsto M_\alpha$ is non decreasing, we derive

$$h^2 \frac{\left(M_3\right)^3}{M_1} \times \frac{1}{M_{-1}M_1} \geq h^2 \ ,$$

Thus, we can see that either

$$h^2 \frac{\left(M_3\right)^3}{M_1} \geq h \text{ or } \frac{1}{M_{-1}M_1} \geq h \ .$$

Hence, Eq. (5) holds.

Since $\mathcal{A}$ is an $(-1, +\infty)$-pattern function, on the one hand M_{-1} and *a fortiori*

M_1 tend toward $+\infty$. On the other hand, from Eq. (3),

$$\frac{h^2 \left(M_3\right)^3}{M_1} \leq (hM_{+\infty})^2 \ .$$

.

Then, since $\lim_{h \to +\infty} hM_{+\infty} = 0$ by hypothesis, we conclude straightforwardly that $L^{\mathrm{NL}}(\mathcal{A}, g, h)$ converges toward $L(g)$. □

In order to include the MDSS based estimators, the hypothesis on the maximal subsegment length, $\lim_{h \to 0} hM_{+\infty} = 0$, should be relaxed. It is replaced in Theorem 6 by a hypothesis on the pattern function distance to the function graph.

Theorem 6. *Let $\mathcal{A}$ be a 1-pattern function. Let $g\colon [a,b] \to \mathbb{R}$ be a concave function whose one-sided derivatives are defined and Lipschitz on $[a,b]$. If, as h tends toward zero, the Hausdorff distance between $\mathcal{D}(g,h)$ and $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$ is bounded*[2]*, then $L^{\mathrm{NL}}(\mathcal{A}, g, h)$ converges toward $L(g)$ and*

$$L(g) - L^{\mathrm{NL}}(\mathcal{A}, g, h) = \mathcal{O}(h) + \mathcal{O}\left(\frac{1}{M_1^{\mathcal{A}}}\right) \ . \tag{6}$$

Proof. Let $h > 0$ and $(a_i)_{i=0}^N = \mathcal{A}(g,h)$. We subdivide each subinterval of the partition $\mathcal{A}(g,h)$ in fixed size segments whose sizes are ℓ and a last segment whose size is not greater than ℓ (we do a sparse estimation of each subinterval). Then, the pattern function $\mathcal{B}$ is defined by $\mathcal{B}(g,h) = (b_i)_{i=0}^{N^{\mathcal{B}}}$ where $b_0 = a_0 = A$ and, for any $i \in [\![1, N^{\mathcal{B}}]\!]$, $b_i = \min\left(b_{i-1} + \ell, a_j\right)$ with $j = \min\{k \mid a_k > b_{i-1}\}$.

Let $k = \max\left\{\left(\mathrm{d}_r g(x) - \mathrm{d}_\ell g(y)\right)/(y - x) \;\middle|\; x < y \in [a,b]\right\}$. From Proposition 2, we have

$$|L(g) - L(g_{\mathrm{c}})| = \mathcal{O}(h) \ . \tag{7}$$

From Proposition 3, we derive

$$\left|L(g_{\mathrm{c}}) - L(g_{\mathrm{c}}^{\mathcal{B}})\right| \leq \sum_{i=1}^{N^{\mathcal{B}}} \frac{k^2}{4} (b_i - b_{i-1})^3 h^3 \leq \frac{k^2}{4} N^{\mathcal{B}} (\ell h)^3 \ ,$$

where

$$N^{\mathcal{B}} = \sum_{i=1}^{N^{\mathcal{A}}} \left\lceil \frac{a_i - a_{i-1}}{\ell} \right\rceil \leq \sum_{i=1}^{N^{\mathcal{A}}} \frac{a_i - a_{i-1}}{\ell} + N^{\mathcal{A}} \leq \frac{B - A}{\ell} + \frac{B - A}{M_1^{\mathcal{A}}} \ .$$

Thus,

$$N^{\mathcal{B}} \leq (b - a) \left(\frac{1}{\ell h} + \frac{1}{h M_1^{\mathcal{A}}}\right) \ . \tag{8}$$

[2] Actually, instead of $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$, we should use the function $x \mapsto \lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor (hx)/h$.

Then

$$\left|L(g_c) - L(g_c^{\mathcal{B}})\right| \leq \frac{k^2}{4}(b-a)\left(\ell^2 h^2 + \frac{\ell^3 h^2}{M_1^{\mathcal{A}}}\right) \ . \tag{9}$$

The functions $\lfloor g_c^{\mathcal{A}} \rfloor$ and $g_c^{\mathcal{B}}$ are piecewise affine. Thus,

$$\begin{aligned}
\|\lfloor g_c^{\mathcal{A}} \rfloor - g_c^{\mathcal{B}}\|_\infty &= \max_{i \in [\![0, N^{\mathcal{B}}]\!]} \left(\left|\lfloor g_c^{\mathcal{A}} \rfloor (hb_i) - g_c^{\mathcal{B}}(hb_i)\right|\right) \\
&\leq \max_{i \in [\![0, N^{\mathcal{B}}]\!]} \left(\left|\lfloor g_c^{\mathcal{A}} \rfloor (hb_i) - h\mathcal{D}(g,h)(b_i)\right|\right) + h \\
&\leq O(h) \quad \text{(from the hypotheses)} \ ,
\end{aligned}$$

Then, the hypotheses of Proposition 4 are satisfied. We derive that there exists two constants U and V, depending on g and $\mathcal{A}$ such that

$$\begin{aligned}
\left|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| &\leq U \sum_{i=1}^{N^{\mathcal{B}}} \frac{h}{(b_i - b_{i-1})} + Vh \\
&\leq U\left((N^{\mathcal{B}} - N^{\mathcal{A}}) \times \frac{h}{\ell} + N^{\mathcal{A}} \times h\right) + Vh \\
&\leq Uh\left(\frac{N^{\mathcal{B}}}{\ell} + N^{\mathcal{A}}\right) + Vh \ .
\end{aligned}$$

Hence, Equation (8) implies

$$\left|L(g_c^{\mathcal{B}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| \leq U(b-a)\left(\frac{1}{\ell^2} + \frac{1}{\ell M_1^{\mathcal{A}}} + \frac{1}{M_1^{\mathcal{A}}}\right) + Vh \ . \tag{10}$$

Eventually, we obtain the following upper bound:

$$\begin{aligned}
&\left|L(g) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| \leq \mathcal{O}(h) + \\
&\frac{k^2}{4}(b-a)\left(\ell^2 h^2 + \frac{\ell^3 h^2}{M_1^{\mathcal{A}}}\right) + U(b-a)\left(\frac{1}{\ell^2} + \frac{1}{\ell M_1^{\mathcal{A}}} + \frac{1}{M_1^{\mathcal{A}}}\right) + Vh \ .
\end{aligned} \tag{11}$$

Taking $\ell = h^{-1/2}$, we obtain the result:

$$\left|L(g) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| = \mathcal{O}(h) + \mathcal{O}(1/M_1^{\mathcal{A}}) \ . \tag{12}$$

Note that, if we assume a uniform distribution of the integers $(a_i - a_{i-1})$ mod ℓ in the interval $[\![0, \ell-1]\!]$, the expected value of $\sum_{i=1}^{N^{\mathcal{B}}} \frac{h}{(b_i - b_{i-1})}$ is in $\mathcal{O}\big((b-a)\big(\frac{1}{\ell^2} + \frac{1}{\ell M_1^{\mathcal{A}}} + \frac{1}{\ell^2 M_1^{\mathcal{A}}}\big)\big)$ for large enough $N^{\mathcal{A}}$. Then, together with $\ell = h^{-1/2}$, Equation (12) becomes $\left|L(g) - L(\lfloor g_c^{\mathcal{A}} \rfloor)\right| = \mathcal{O}(h) + \mathcal{O}(h^{1/2}/M_1^{\mathcal{A}})$. □

On our example with the logarithm, the observed error for the MDSS method (see Figure 3) is in $\mathcal{O}(h)$ which is better than the expected convergence rate $\mathcal{O}(h) + \mathcal{O}(h^{1/2}/M_1^{\mathcal{A}})$ (and *a fortiori* better than the worst case convergence rate $\mathcal{O}(h) + \mathcal{O}(1/M_1^{\mathcal{A}})$). Indeed, the mean M_1 for the MDSS pattern function lies

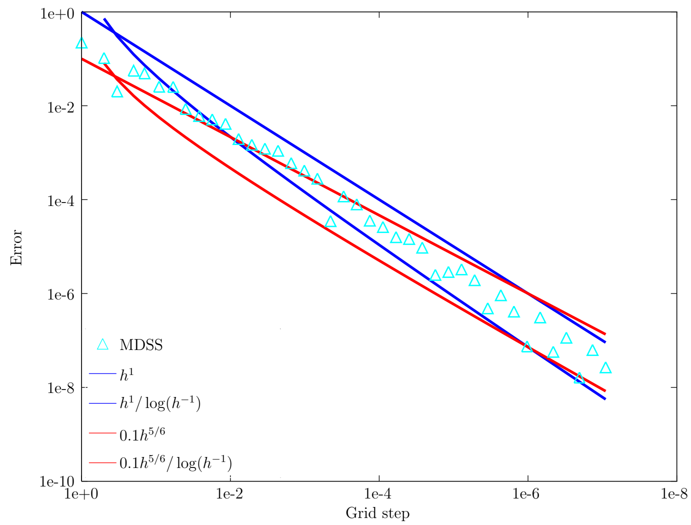

Figure 3: $\left|L(g) - L(\lfloor g_c^{\mathrm{MDSS}} \rfloor)\right|$. The continuous lines correspond to the convergence rates derived from Theorem 6 and Theorem 9 (see text).

between $\mathcal{O}(h^{-1/3})$ and $\mathcal{O}(h^{-1/3}\log(h^{-1}))$ [5], so the bound for the expected convergence rate lies between $O(h^{5/6})$ and $O(h^{5/6}\log(h^{-1}))$.

In the next section, we introduce the notion of *biconcavity* which corresponds to the actual behavior of MDSS and we show that this property speeds up the convergence rate and explains the observed convergence rate of the MDSSE.

4. Biconcavity

When the function g is concave, the piecewise affine function $g_c^{\mathcal{A}}$ is clearly also concave. Nevertheless, the second piecewise function $\lfloor g_c^{\mathcal{A}} \rfloor$ is not necessarily concave. When, below some threshold h_0, the function $\lfloor g_c^{\mathcal{A}} \rfloor$ is concave for any $h > 0$, we say that g is *biconcave relative to* $\mathcal{A}$. In Appendix A.2, we exhibit a concave function that is not biconcave relative to any local estimator. Nevertheless, it follows from the very definition of $\lfloor g_c^{\mathcal{A}} \rfloor$ that its hypograph is *digitally convex* (the convex hull of the hypograph does not contain more integer points than the hypograph itself) and it was proved in [6] that the MDSS of the boundary of digitally convex body of $\mathbb{Z}^2$ are monotonic. Hence, continuous concave functions are biconcave relative to the MDSSE pattern function.

This section gives a sufficient condition to get the biconcavity property and studies the consequences on the convergence speed of such a property.

Proposition 7. *Let $\mathcal{A}$ be pattern function and let $g\colon [a,b] \to \mathbb{R}$ be a concave function such that, for some constant $k > 0$, it is true that* $\mathrm{d}_r g(x) - \mathrm{d}_\ell g(y) \geq$

$k(y-x)$ for any $x, y \in [a,b]$ such that $x < y$. If one of the following conditions holds, then the piecewise affine function $\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor$ is concave.

(i) $h{M_{-\infty}}^2 \geq 2/k$,

(ii) $h{M_{-\infty}}^2 \geq 1/k$ *and* $\mathcal{A}(g,h)$ *is a constant sequence.*

Proof. Let $\delta_i = a_i - a_{i-1}$ for $1 \leq i \leq N$. The piecewise affine function $\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor$ is concave iff, for any $i \in [\![1, N-1]\!]$,

$$\frac{\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_{i+1}) - \left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_i)}{h\delta_{i+1}} \leq \frac{\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_i) - \left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_{i-1})}{h\delta_i} \quad . \tag{13}$$

Since, for any $k \in [\![0, N]\!]$, $\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_k)$ is a multiple of h, Equation (13) can be rewritten as

$$\delta_i\Big(\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_{i+1}) - \left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_i)\Big) - \delta_{i+1}\Big(\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_i) - \left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_{i-1})\Big) < h \gcd(\delta_i, \delta_{i+1}).$$

Thus, from the very definition of the function $\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor$, we derive that Equation (13) is true whenever

$$\delta_i\big(g(ha_{i+1}) - g(ha_i) + h\big) - \delta_{i+1}\big(g(ha_i) - g(ha_{i-1}) - h\big) \leq h \gcd(\delta_i, \delta_{i+1}). \tag{14}$$

Now, from the hypotheses, we derive that, for any $x, y \in [a,b]$ such that $x < y$,

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(t)\,\mathrm{d}t \\ &\leq \int_x^y \mathrm{d}_r g(x) - k(t-x)\,\mathrm{d}t \\ &\leq \mathrm{d}_r g(x)(y-x) - \frac{1}{2}k(y-x)^2 \quad . \end{aligned}$$

Alike,

$$\mathrm{d}_\ell g(y)(y-x) + \frac{1}{2}k(y-x)^2 \leq g(y) - g(x) \quad .$$

Then

$$g(ha_{i+1}) - g(ha_i) \leq \mathrm{d}_r g(ha_i)h\delta_{i+1} - \frac{1}{2}k(h\delta_{i+1})^2$$

and

$$\mathrm{d}_\ell g(ha_i)h\delta_i + \frac{1}{2}k(h\delta_i)^2 \leq g(ha_i) - g(ha_{i-1})$$

Thus, Equation (14) is true whenever

$$h\delta_i\delta_{i+1}\Big(\mathrm{d}_r g(ha_i) - \frac{1}{2}kh\delta_{i+1} - \mathrm{d}_\ell g(ha_i) - \frac{1}{2}kh\delta_i\Big) \leq h\Big(\gcd(\delta_i, \delta_{i+1}) - \delta_i - \delta_{i+1}\Big) \quad .$$

Noting that $\mathrm{d}_r g(ha_i) \leq \mathrm{d}_\ell g(ha_i)$, we get the following sufficient inequality

$$h(M_{-\infty})^2 k(\delta_{i+1} + \delta_i) \geq 2\big(\delta_i + \delta_{i+1} - \gcd(\delta_i, \delta_{i+1})\big) \ .$$

That is

$$h(M_{-\infty})^2 k \geq 2\big(1 - \frac{\gcd(\delta_i, \delta_{i+1})}{\delta_{i+1} + \delta_i}\big) \ .$$

Proposition 7 follows straightforwardly. □

The next proposition is an improvement of Proposition 4 in case of biconcavity. It is a consequence of Lemma 15.

Proposition 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two pattern functions such that $\mathcal{B} \prec \mathcal{A}$, $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$ is concave and $\| \lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor - \lfloor g_{\mathrm{c}}^{\mathcal{B}} \rfloor \|_\infty \leq \omega h$ for some $\omega > 0$. Then*

$$\big| L(g_{\mathrm{c}}^{\mathcal{B}}) - L(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor) \big| \leq U h \ , \tag{15}$$

where $U = \max(\alpha, \alpha - 2\beta)$ with $\alpha = \varphi'(g'(a) + 1)$ and $\beta = \varphi'(g'(b) - 1)$.

Proof. From the hypotheses, we have

$$\big(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor - \omega h \big) \leq g_{\mathrm{c}}^{\mathcal{B}} \leq \big(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor - \omega h \big) + (2\omega + 1)h \ .$$

Moreover, $g_{\mathrm{c}}^{\mathcal{B}}$ is concave (for g is concave).
Let $s_1^{\mathcal{A}}$ and $s_2^{\mathcal{A}}$, resp. $s_1^{\mathcal{B}}$ and $s_2^{\mathcal{B}}$, be the slopes of the first and last segments of $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$, resp. $g_{\mathrm{c}}^{\mathcal{B}}$. From Lemma 15, applied with $f_1 = \lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor - \omega h$, $f_2 = g_{\mathrm{c}}^{\mathcal{B}}$ and $e = (2\omega + 1)h$, we derive

$$\big| L(g_{\mathrm{c}}^{\mathcal{B}}) - L(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor) \big| \leq U_0 h \ ,$$

where $U_0 = \max(\varphi'(s_1), \varphi'(s_1) - 2\varphi'(s_2))$ with s_i, $i \in \{1, 2\}$, lying between $s_i^{\mathcal{A}}$ and $s_i^{\mathcal{B}}$.
Let $(a_i)_{i=0}^N = \mathcal{A}(g, h)$, $\delta_1 = a_1 - a_0$ and $\delta_N = a_N - a_{N-1}$. It can easily be seen that

$$s_1^{\mathcal{A}} < s_1^{\mathcal{B}} + 1/\delta_1$$

and

$$s_2^{\mathcal{A}} > s_2^{\mathcal{B}} - 1/\delta_N \ .$$

Then, since g is concave,

$$s_1^{\mathcal{A}} < g'(a) + 1/\delta_1 \leq g'(a) + 1$$

and

$$s_2^{\mathcal{A}} > g'(b) - 1/\delta_N \geq g'(b) - 1 \ .$$

Thus,

$$s_1 \leq \max(s_1^{\mathcal{A}}, s_1^{\mathcal{B}}) < g'(a) + 1$$

and

$$s_2 \geq \min(s_2^{\mathcal{A}}, s_2^{\mathcal{B}}) > g'(b) - 1 \ .$$

As the function φ' is increasing, we get

$$\varphi'(s_1) < \alpha$$

and

$$\varphi'(s_2) > \beta \ .$$

then

$$U_0 < U$$

and the result holds.

□

The following theorem is the consequence of Proposition 8 on the convergence speed of the non-local estimators.

Theorem 9. *Let $\mathcal{A}$ be a 1-pattern function. Let $g\colon [a,b] \to \mathbb{R}$ be a biconcave function relative to $\mathcal{A}$ whose one-sided derivatives are defined and Lipschitz on $[a,b]$. If, as h tends toward zero, the Hausdorff distance between $\mathcal{D}(g,h)$ and $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$ is bounded, then*

$$L(g) - L^{\mathrm{NL}}(g,h) = \mathcal{O}(h) + \mathcal{O}\left(\frac{h^{2/3}}{M_1^{\mathcal{A}}}\right) \ .$$

Proof. The proof is similar to the proof of Theorem 6 except that we invoke Proposition 8 instead of Proposition 4. Then, in Equation (10), the term $(b-a)\left(\frac{1}{\ell^2} + \frac{1}{\ell M_1^{\mathcal{A}}} + \frac{1}{M_1^{\mathcal{A}}}\right)$ vanishes and we get

$$\left|L(g) - L(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor)\right| \leq \mathcal{O}(h) + \frac{k^2}{4}\left(\ell^2 h^2 + \frac{\ell^3 h^2}{M_1^{\mathcal{A}}}\right) \ .$$

Taking $\ell = h^{-4/9}$, we obtain the result:

$$\left|L(g) - L(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor)\right| = \mathcal{O}(h) + \mathcal{O}\left(\frac{h^{2/3}}{M_1^{\mathcal{A}}}\right) \ .$$

□

Observe that, for the MDSS pattern function on the set of C^3 functions with positive curvature, we have ([5]) $\Omega(h^{-1/3}) \leq M_1 \leq \mathcal{O}(h^{-1/3}\log(h^{-1}))$. Then

$$\mathcal{O}\left(\frac{h}{\log(h^{-1})}\right) \leq \left|L(g) - L(\lfloor g_{\mathrm{c}}^{\mathrm{MDSS}} \rfloor)\right| \leq \mathcal{O}(h) \ . \tag{16}$$

Equation 16 fits the MDSS convergence rates reported in Figure 3.

5. Conclusion

In this paper, thanks to the concavity assumption, we improve previous results on the multigrid convergence rate of the Non Local Estimators, a class of estimators that relies on a polygonal interpolation of the continuous function digitization. Furthermore, we introduce the notion of *biconcavity* which is satisfied by the MDSS estimator and by the sparse estimators with enough large pattern sizes. Biconcavity allows further improvement of the convergence rate, up to $\mathcal{O}(h)$ in the worst case, which is optimal with a square grid whose step is h. The proposed tests give convergence rates corresponding to the theoretical ones.

Besides, some preliminary experiments indicate that the convergence rates for concave functions also apply to a wide class of neither concave nor convex functions. The test is the following: The discretization and the quantization errors are measured for some function-graph length-estimation with respect to the resolution $r = 1/h$. The NLE pattern function generates random steps uniformly distributed between $0.5h^{-1/2}$ and $1.5h^{-1/2}$. Then, both error upper bounds for concave functions (Prop. 3 and Prop. 4) are in $\mathcal{O}(h)$. The function f_0 is a concave function ($f_0(x) = \ln(x)$, $x \in [1, 2]$) and the other functions are defined as follows: $f_i(x) = f_0(x) + P_i(x)$, $i \in [1, 4]$, where P_i is a trigonometric polynomial. The polynomials P_i, $i \in \{1, 2\}$ are randomly generated as follows:

$$P_i(x) = \sum_{j=1}^{10} \frac{a_{i,j}}{(2\pi f_{i,j})^i} \sin(2\pi f_{i,j} x + \varphi_{i,j})$$

where $a_{i,j} \in [1, 10]$, $f_{i,j} \in [2^j, 2^{j+1}]$ and $\varphi_{i,j} \in [0, 2\pi)$. The polynomial P_3 is the sine of P_1 with the highest frequency ($f_{1,10} = 1719$) and $P_4(x) = P_3(x)/30$. The relative magnitudes of the P_i and their first two derivatives with respect to those of f_0 are gathered in Table 1.

i	1	2	3	4
P_i	50%	1.5%	0.07%	0.05%
P_i'	4000%	30%	500%	1%
P_i''	10^7%	3000%	5.10^6%	100%

Table 1: Relative magnitudes of the trigonometric polynomials P_i and their first two derivatives with respect to those of f_0.

From the length estimation convergence rates shown in Fig. 4, it seems that curves with finitely many inflection points behave like concave or convex curves above some resolution. It is also possible that a combination of Th. 5 and Th. 6 would apply on curves with bounded curvatures. This very first test shows the necessity to deepen the research on this subject.

The NLE framework with its pattern functions appears to be an efficient tool to study the multigrid convergence of the length estimators. Future works will extend to the plane curves the obtained results and prospect the relaxation of

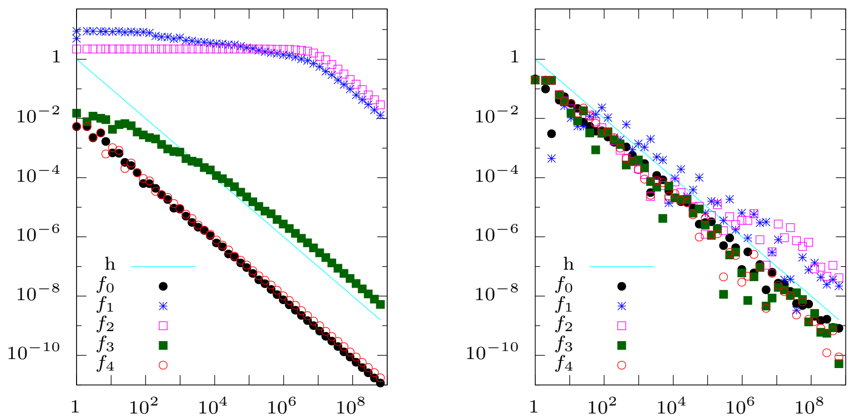

Figure 4: The discretization error (left) and the quantization error (right) with respect to the resolution $r = 1/h$ for a concave function ($f_0(x) = \ln(x)$, $x \in [1, 2]$) and four functions $f_i(x) = f_0(x) + P_i(x)$, $i \in [1, 4]$, where P_i is a trigonometric polynomial (see text).

the concavity assumption. Also, they should investigate more finely the behavior of the quantization error.

Appendix A. Counterexamples

Appendix A.1. An inferior bound for the convergence speed of a concave function

We present in this section an example of a parabola rectification by a sparse estimator where the bound found in Theorem 5 is reached.

Let $H = h^{-\gamma}$ with $0 < \gamma < 1$ be the step of the sparse estimator, the pattern function of which is noted $\mathcal{A}$ ($\mathcal{A}$ is a (α, β)-pattern function for any α, β in $\overline{\mathbb{R}} \setminus \{0\}$). Let g be the function defined on the interval $I = [\frac{1}{16}, \frac{19}{48}]$ by $g(x) = (\frac{19}{48})^2 - x^2$. The function g clearly satisfies the hypotheses of Theorem 5 and the k-th power mean $M_k^{\mathcal{A}}$ is in $\mathcal{O}(h^{-\gamma})$ for any non-zero real number k. Then, from Theorem 5 we get

$$L(g) - L^{\mathrm{NL}}(\mathcal{A}, g, h) = \mathcal{O}(h^{2(1-\gamma)}) + \mathcal{O}(h^{2\gamma}) \ .$$

Thereby, the best choice for H is $h^{-1/2}$ which gives $L(g) - L^{\mathrm{NL}}(\mathcal{A}, g, h) = \mathcal{O}(h)$. Let $g_{\mathrm{c}}^{\mathcal{A}}$ and $\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor$ be the piecewise affine functions defined in Section 2.2. Then, we shall prove below that the lengths of their curves satisfy $L(\lfloor g_{\mathrm{c}}^{\mathcal{A}} \rfloor) + 0.07h \leq L(g_{\mathrm{c}}^{\mathcal{A}}) \leq L(g)$ for any $h = (12(8p+1))^{-2}$ where $p \in \mathbb{N}$. Observe that the bounds of the interval I are multiple of h. Hence, there is no error due to the bounds (*i.e.* $g_{\mathrm{c}}^{\mathcal{A}} = g$). Moreover, the function g verifies the condition *(i)* of Prop. 7 and is then biconcave relative to $\mathcal{A}$. Eventually, for any $p \in \mathbb{N}$ and $h = (12(8p+1))^{-2}$, we get $L(g) - L^{\mathrm{NL}}(\mathcal{A}, g, h) \geq 0.07h$ which proves that the convergence rate in Theorem 5 cannot be improved in the general case.

Detailed calculus.
The notations are those introduced in Paragraph *Notations* of Section 2.2.
Let $h = \frac{1}{144(8p+1)^2}$ ($p \in \mathbb{N}$) and $H = h^{-\frac{1}{2}} = 12(8p+1)$.
Thereby, here we have

$$\begin{aligned} A &= 9(8p+1)^2 \text{ and } Ah = \frac{1}{16} \ , \\ B &= 57(8p+1)^2 \text{ and } Bh = \frac{19}{48} \ , \\ N &= \left\lceil \frac{\frac{19}{48} - \frac{1}{16}}{hH} \right\rceil = 4(8p+1) \ , \\ \forall i \in [\![0, N]\!], \quad ha_i &= \frac{1}{16} + ihH = \frac{1}{16} + i\sqrt{h} \ . \end{aligned}$$

313 Furthermore, we have

$$g(ha_i) = \left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor (ha_i) + (i \mod 2) \times \frac{h}{2} \ . \tag{A.1}$$

We also set

$$\begin{aligned} c &= \frac{h}{2} \ , \\ z_i &= h\frac{(a_i + a_{i+1})}{2} \ , \\ y_i &= g(ha_{i+1}) - g(ha_i) \\ &= -2\sqrt{h}\, z_i \ . \end{aligned}$$

Then, from (A.1), we derive

$$\begin{aligned} L(g_{\mathrm{c}}^{\mathcal{A}}) - L(\left\lfloor g_{\mathrm{c}}^{\mathcal{A}} \right\rfloor) = \sum_{i=0}^{N/2-1} & \left(\sqrt{h + {y_{2i}}^2} + \sqrt{h + {y_{2i+1}}^2} \right) \\ & - \left(\sqrt{h + (y_{2i} - c)^2} + \sqrt{h + (y_{2i+1} + c)^2} \right) \ . \end{aligned}$$

On the one hand

$$\begin{aligned} \sqrt{h + {y_{2i}}^2} - \sqrt{h + (y_{2i} - c)^2} &= -\frac{h}{4} \frac{8z_{2i} + \sqrt{h}}{\sqrt{1 + 4{z_{2i}}^2} + \sqrt{1 + 4(z_{2i} + \frac{1}{4}\sqrt{h})^2}} \\ &\geq -\frac{h}{8} \frac{8z_{2i} + \sqrt{h}}{\sqrt{1 + 4{z_{2i}}^2}} \ . \end{aligned}$$

On the other hand

$$\begin{aligned} \sqrt{h + {y_{2i+1}}^2} - \sqrt{h + (y_{2i+1} + c)^2} &= \frac{h}{4} \frac{8z_{2i+1} - \sqrt{h}}{\sqrt{1 + 4{z_{2i+1}}^2} + \sqrt{1 + 4(z_{2i+1} - \frac{1}{4}\sqrt{h})^2}} \\ &\geq \frac{h}{8} \frac{8z_{2i+1} - \sqrt{h}}{\sqrt{1 + 4{z_{2i+1}}^2}} \ . \end{aligned}$$

By summing,

$$L(g_c^{\mathcal{A}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor) \geq$$
$$h \sum_{i=0}^{16p+1} \left(\frac{z_{2i+1}}{\sqrt{1+4{z_{2i+1}}^2}} - \frac{z_{2i}}{\sqrt{1+4{z_{2i}}^2}} \right) - \frac{h\sqrt{h}}{8} \sum_{i=0}^{32p+3} \frac{1}{\sqrt{1+4{z_i}^2}} \quad .$$

Since the function $f_1(x) = \frac{x}{\sqrt{1+4x^2}}$ is monotonically increasing and concave, one has

$$\begin{aligned} \sum_{i=0}^{16p+1} \left(f_1(z_{2i+1}) - f_1(z_{2i})\right) &\geq \frac{1}{2} \sum_{i=0}^{32p+3} \left(f_1(z_{i+1}) - f_1(z_i)\right) \\ &\geq \frac{1}{2} \left(f_1(z_{32p+4}) - f_1(z_0)\right) \quad . \end{aligned}$$

Moreover, the function $f_2(x) = \frac{1}{\sqrt{1+4x^2}}$ is monotonically decreasing and convex. Thus the Riemann sum $\sum_{i=0}^{32p+3} \frac{1}{\sqrt{1+4{z_i}^2}} \times \sqrt{h}$ is bounded by the integral $\int_{\frac{1}{16}}^{\frac{19}{48}} f_2(x)\,\mathrm{d}x$. It follows that

$$\begin{aligned} L(g_c^{\mathcal{A}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor) \geq \frac{h}{2} \Bigg(& f_1\big(\frac{19}{48} + \frac{\sqrt{h}}{2}\big) - f_1\big(\frac{1}{16} + \frac{\sqrt{h}}{2}\big) \\ & -\frac{1}{8} \operatorname{arg\,sinh}\big(\frac{19}{24}\big) + \frac{1}{8} \operatorname{arg\,sinh}\big(\frac{1}{8}\big) \Bigg) \quad . \end{aligned}$$

Since $\sqrt{h} \leq \frac{1}{12}$ for any $p \in \mathbb{N}$, we obtain

$$L(g_c^{\mathcal{A}}) - L(\lfloor g_c^{\mathcal{A}} \rfloor) > 0.076h.$$

Eventually, for any $h = \frac{1}{(12(8p+1))^2}$, we have shown that

$$L(g) \geq L(g_c^{\mathcal{A}}) \geq L(\lfloor g_c^{\mathcal{A}} \rfloor) + 0.07h.$$

This example shows that for some non-local estimators, the obtained bounds are tight and therefore cannot be improved in the general case.

Appendix A.2. Biconcavity

In this section, we exhibit a concave function whose discretizations contain arbitrary long convex pairs of chords. The counterexample relies on the following theorem proved in [7]. This theorem asserts that, given a function $x \mapsto ax^2 + bx + c$, the distribution in $[0, h]$ of the values of the expression $\big(a(kh)^2 + b(kh) + c\big) \bmod h$, $k \in \mathbb{N}$, which are the errors resulting from the quantization in $h\mathbb{Z}$, tends toward the equidistribution.

Theorem 10 ([7, Lemma 2 and Prop. 3]). *Let $a, b \in \mathbb{R}$, $a < b$. Let $g : [a,b] \to \mathbb{R}$ be a polynomial function of degree 2. Then, for all interval $I \subseteq [0,1]$,*

$$\lim_{h \to 0} \frac{\mathrm{card}\{x \in h\mathbb{Z} \cap [a,b] \mid g(x) \bmod h \in hI\}}{\mathrm{card}(h\mathbb{Z} \cap [a,b])} = \mu(I) \ ,$$

where $\mu(I)$ is the classical length of I.

Let us consider the function $g(x) = 2x - x^2$, $x \in [0,1]$, which is concave. We denote by $\lfloor g \rfloor_h$ the function $x \in [0,1] \mapsto \lfloor g(x)/h \rfloor h \in h\mathbb{Z}$. Let H be a positive integer. Thanks to Theorem 10, we prove that, for each grid spacing h below some threshold, we can choose an integer p such that the finite difference $\lfloor g \rfloor_h((p+H)h) - \lfloor g \rfloor_h(ph)$ is less than or equal to the grid spacing h while the finite difference $\lfloor g \rfloor_h((p+2)Hh) - \lfloor g \rfloor_h(ph)$ is greater than twice the grid spacing h. Thus, the graph of $\lfloor g \rfloor_h$ has a convex pair of consecutive chords.

Detailed calculus.
According to Theorem 10 with $[a,b] = [1 - \frac{17}{24H}, 1 - \frac{16}{24H}]$ and $I = [\frac{4}{12}, \frac{7}{12})$, it exists a real $h_0 > 0$ such that, for any $h \in (0, h_0)$, one has

$$\mathrm{card}\left\{n \in J \mid g(nh) - \lfloor g \rfloor_h(nh) \in [\tfrac{4h}{12}, \tfrac{7h}{12})\right\} \geq \frac{1}{5}\,\mathrm{card}\, J \ ,$$

where $J = [\![\frac{a}{h}, \frac{b}{h}]\!]$.

Since $\mathrm{card}\, J \to +\infty$ as $h \to 0$, there exists $h_1 > 0$ such that for any $h < h_1$, one can find $n_0 \in \mathbb{N}$ such that $[\![n_0 H, (n_0+2)H]\!] \subset J$ and $g(n_0 hH) - \lfloor g \rfloor_h(n_0 hH) \in [\frac{4h}{12}, \frac{7h}{12})$.

Let $h < h_1$. Noting that $\frac{16}{12H} \leq g'(x) \leq \frac{17}{12H}$ on $[a,b]$, we claim that

$$\begin{aligned}
\lfloor g \rfloor_h((n_0+1)hH) - &\lfloor g \rfloor_h(n_0 hH) \\
&< g((n_0+1)hH) - (g(n_0 hH) - \frac{7}{12}h) \\
&< \frac{17}{12H} \times hH + \frac{7}{12}h \\
&< 2h \ .
\end{aligned}$$

As the left hand side of the above inequalities is a multiple of h, we get

$$\lfloor g \rfloor_h((n_0+1)hH) - \lfloor g \rfloor_h(n_0 hH) \leq h \ .$$

In the same way, we obtain

$$\begin{aligned}
\lfloor g \rfloor_h((n_0+2)hH) - \lfloor g \rfloor_h(n_0 hH)& \\
&> g((n_0+2)hH) - h - (g(n_0 hH) - \frac{4}{12}h) \\
&> \frac{16}{12H} \times 2hH - \frac{2}{3}h \\
&> 2h \ .
\end{aligned}$$

Thus,

$$\lfloor g \rfloor((n_0+2)hH) - \lfloor g \rfloor(n_0hH) \geq 3h \ .$$

Finally, we have

$$\lfloor g \rfloor((n_0+2)hH) - \lfloor g \rfloor(n_0hH) > 2\left(\lfloor g \rfloor((n_0+1)hH) - \lfloor g \rfloor(n_0hH)\right) \ .$$

That is, the function $\lfloor g \rfloor$ is strictly convex on $[n_0hH, (n_0+2)hH]$.

Appendix B. Technical lemmas

Lemma 11. *Let f be a Lipschitz continuous function defined on an interval $[a,b]$. Let m, M be two real numbers such that $m \leq f'(t) \leq M$ for any $t \in [a,b]$ where the derivative of f is defined. Then, the length $L(f)$ of the graph of f is less than, or equal to, the length of the polylines joining the points $A(a, f(a))$ and $B(b, f(b))$ with segments of slopes m or M.*

Proof. We assume without loss of generality that $[a,b] = [0,1]$. Let s be the slope of the line from A to B. Since f is Lipschitz continuous, it is almost everywhere differentiable and the slope s is equal to the integral of f' on $[0,1]$. Thus, $m \leq s \leq M$ and there exists $k \in [0,1]$ such that $s = (1-k)m + kM$. Moreover,

$$L(f) = \int_0^1 \varphi \circ f'(t)\,\mathrm{d}t.$$

and it can easily be seen that the length of any polyline joining the points A and B with segments of slopes m or M is $L = (1-k)\varphi(m) + k\varphi(M)$.

We shall prove that $L(f) - s \leq L - s$, that is

$$\int_0^1 \psi \circ f'(t)\,\mathrm{d}t \leq (1-k)\psi(m) + k\psi(M) \ ,$$

where $\psi(x) = \varphi(x) - x$. Observe that the function ψ is positive, decreasing and convex.

Let $\psi \circ g$ be a simple function such that $0 < \psi \circ g \leq \psi \circ f'$ (since ψ is bijective from $\mathbb{R}$ to $]0, +\infty[$, any positive simple function can be written as $\psi \circ g$). From $\psi \circ g \leq \psi \circ f'$, we derive that $g \geq f'$. Thus, $g \geq m$. Furthermore, even if it means replacing g by $\inf(g, M)$, we may assume that $g \leq M$. Now, let k_1 be the real in $[0,1]$ such that

$$\int_0^1 g(t)\,\mathrm{d}t = (1-k_1)m + k_1M \ .$$

As $g \geq f'$, we have $k_1 \geq k$ and, since ψ is convex and decreasing,

$$\int_0^1 \psi \circ g(t)\,\mathrm{d}t \leq (1-k_1)\psi(m) + k_1\psi(M) \leq (1-k)\psi(m) + k\psi(M) \ . \quad \text{(B.1)}$$

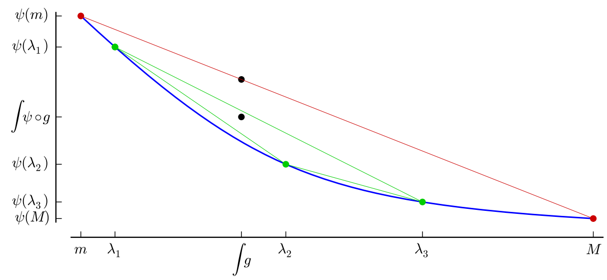

Figure B.5: An illustration of the first inequality in (B.1). We assume $g = \sum_{i=0}^{n} \lambda_i 1_{E_i}$ where, for any i, $m \leq \lambda_i \leq M$, the measurable sets E_i are pairwise disjoint and $\sum_{i=0}^{n} \mu(E_i) = 1$ (here, μ is the Lebesgue measure on $\mathbb{R}$). Thus, the point with coordinates $(\int g, \int \psi \circ g)$ is the barycenter of the weighted points $\big((\lambda_i, \psi(\lambda_i)), \mu(E_i)\big)$ while the point with coordinates $(\int g, (1-k_1)\psi(m) + k_1\psi(M))$ is the barycenter of the weighted points $\big((m, \psi(m)), 1-k_1\big)$, $\big((M, \psi(M)), k_1\big)$.

The first inequality in Equation B.1 is illustrated, and commented, in Figure B.5.

Eventually,

$$\int_0^1 \psi \circ f'(t)\,\mathrm{d}t = \max_g \int_0^1 \psi \circ g(t)\,\mathrm{d}t \leq (1-k)\psi(m) + k\psi(M) \ .$$

□

Lemma 12. *Let ABC be a triangle in $\mathbb{R}^2$ ($A \neq C$) with edges of slopes $-\infty < \alpha < \beta < \gamma < +\infty$. We assume that the edge AC have slope β. Then,*

$$\frac{AB + BC - AC}{AC} \leq \frac{(\gamma - \alpha)^2}{4\varphi(\beta)} \ .$$

Fig. B.6 illustrates the configuration studied in Lemma 12.

Proof. Let $k \in (0,1)$ such that $\beta = k\gamma + (1-k)\alpha$. Let m be the abscissa of **AC**. It can be seen that the vectors **AB**, **BC** and **AC** have coordinates $(km, km\gamma)$, $((1-k)m, (1-k)m\alpha)$ and $(m, m\beta)$. Thus,

$$\begin{aligned} AB + BC - AC &= m\big(k\varphi(\gamma) + (1-k)\varphi(\alpha) - \varphi(\beta)\big) \\ &= m\Big(k\big(\varphi(\gamma) - \varphi(k\gamma + (1-k)\alpha)\big) + \\ &\qquad (1-k)\big(\varphi(\alpha) - \varphi(k\gamma + (1-k)\alpha)\big)\Big) \\ &= mk(1-k)(\gamma - \alpha)\big(\varphi'(\xi_1) - \varphi'(\xi_2)\big) \\ &= mk(1-k)(\gamma - \alpha)(\xi_1 - \xi_2)\varphi''(\xi) \ , \end{aligned}$$

where ξ_1, ξ_2, ξ lie between α and γ.

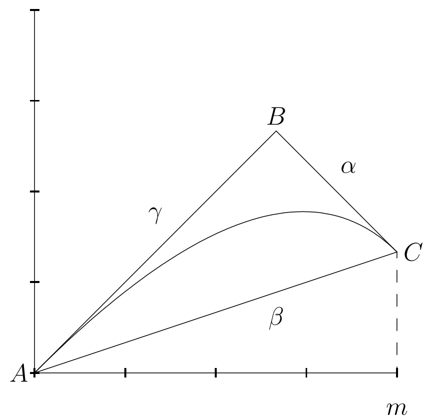

Figure B.6: α, β, γ are the slopes of the segments BC, CA, AB.

Hence,

$$AB + BC - AC \leq \frac{m(\gamma-\alpha)^2}{4} \ , \tag{B.2}$$

for $\|\varphi''\|_\infty = 1$. As $AC = m\varphi(\beta)$, the result holds. □

Lemma 13. *Let $(u_n)_{n\in\mathbb{N}}$ a monotonically non-increasing sequence of real non negative numbers and $(c_n)_{n\in\mathbb{N}}$ a sequence of reals in an interval I such that $\sum_{i=0}^{j} c_i \in I$ for any integer j. Then, $\sum_{i=0}^{j} c_i\, u_i \in u_0\, I$ for any integer j.*

Proof. If $u_0 = 0$, then $u_n = 0$ for any n and the result is obvious. From now, we assume $u_0 > 0$. Let $n \in \mathbb{N}$ and $S = \sum_{i=0}^{n} c_i\, u_i$. We set $C_j = \sum_{i=0}^{j} c_i$ for any $j \leq n$, $p_i = \frac{u_i - u_{i+1}}{u_0}$ for any $i \leq n-1$ and $p_n = \frac{u_n}{u_0}$. The reals p_i are all non-negative and their sum equals 1. We can easily check that

$$\begin{aligned} S &= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} c_j \right) (u_i - u_{i+1}) + \left(\sum_{j=0}^{n} c_j \right) u_n \\ &= u_0 \left(\sum_{i=0}^{n} p_i\, C_i \right) \ . \end{aligned}$$

The last equality above shows that the real $\frac{1}{u_0} S$ is the barycenter –with non-negative weights– of numbers in the interval I. Thus, the result holds. □

Lemma 14. *Let f_1 and f_2 be two piecewise affine functions defined on $[c,d] \subset \mathbb{R}, (c < d)$, with a common partition $\sigma = (x_i)_{i=0}^{p}$ having p steps and such that $f_1 \leq f_2 \leq f_1 + e$ for some constant $e > 0$. If furthermore f_2 is concave, then*

$$\begin{aligned} |L(f_1) - L(f_2)| &\leq \sum_{i=1}^{p} \frac{1}{x_i - x_{i-1}} e^2 + U e \\ &\leq \frac{p}{M_{-1}(\sigma)} e^2 + U e \ . \end{aligned}$$

where $U = \max(\varphi'(s_{2,0}), \varphi'(s_{2,0}) - 2\varphi'(s_{2,p-1}))$ is a constant which depends on the slopes $s_{2,0}$ and $s_{2,p-1}$ of the first and the last segments of f_2.

PROOF. Let $\sigma = (x_i)_{i=0}^{p}$ be the common partition for f_1 and f_2. We write m_i for $x_{i+1} - x_i$ and $s_{1,i}$, resp. $s_{2,i}$, for the slope of f_1, resp. f_2, on the interval $[x_i, x_{i+1}]$. Then,

$$\begin{aligned}
L(f_1) - L(f_2) &= \sum_{i=0}^{p-1} m_i\big(\varphi(s_{1,i}) - \varphi(s_{2,i})\big) \\
&= \sum_{i=0}^{p-1} \varphi'(s_{0,i})\, m_i(s_{1,i} - s_{2,i}) \quad \text{where } s_{0,i} \in [s_{1,i}, s_{2,i}] \\
&= \sum_{i=0}^{p-1} \varphi'(s_{2,i})\, m_i(s_{1,i} - s_{2,i}) + \\
&\qquad\qquad \sum_{i=0}^{p-1} (\varphi'(s_{0,i}) - \varphi'(s_{2,i}))\, m_i(s_{1,i} - s_{2,i}) \ .
\end{aligned}$$

Let give an upper bound for C= $\left|\sum_{i=0}^{p-1} \varphi'(s_{2,i})\, m_i(s_{1,i} - s_{2,i})\right|$. Since the function f_2 is concave, the sequence $(s_{2,i})_{i=0}^{p-1}$ is non-increasing as is the sequence $(\varphi'(s_{2,i}))_{i=0}^{p-1}$ (for the function φ' is increasing). Hence, we can apply Lemma 13 with the settings

$$\begin{aligned}
c_i &= m_i(s_{1,i} - s_{2,i}) \\
&= (f_1(x_{i+1}) - f_2(x_{i+1})) - (f_1(x_i) - f_2(x_i)) \ , \\
u_i &= \varphi'(s_{2,i}) - \varphi'(s_{2,p-1}) \ , \\
I &= [-e, e] \ .
\end{aligned}$$

Lemma 13 induces that $\left|\sum_{i=0}^{p-1} u_i c_i\right| \leq u_0 e$. Then, we get

$$\begin{aligned}
\mathrm{C} &\leq \left|\sum_{i=0}^{p-1} u_i c_i\right| + \left|\sum_{i=0}^{p-1} \varphi'(s_{2,p-1}) c_i\right| \\
&\leq u_0 e + |\varphi'(s_{2,p-1})| \, \big|\big(f_1(d) - f_2(d)\big) - \big(f_1(c) - f_2(c)\big)\big| \\
&\leq u_0 e + |\varphi'(s_{2,p-1})| \, e \\
&\leq U e \ ,
\end{aligned}$$

where $U = \max(\varphi'(s_{2,0}), \varphi'(s_{2,0}) - 2\varphi'(s_{2,p-1}))$.

We now look at the sum D= $\sum_{i=0}^{p-1} (\varphi'(s_{0,i}) - \varphi'(s_{2,i}))\, m_i(s_{1,i} - s_{2,i})$. The function φ' is 1-Lipschitz ($\varphi''(x) = (1 + x^2)^{(-3/2)}$), so we have

$$|\varphi'(s_{0,i}) - \varphi'(s_{2,i})| \leq |s_{0,i} - s_{2,i}| \leq |s_{1,i} - s_{2,i}| \ .$$

Then,

$$\mathrm{D} \le \sum_{i=0}^{p-1} m_i (s_{1,i} - s_{2,i})^2 \le \sum_{i=0}^{p-1} \frac{c_i^2}{m_i} \le \sum_{i=0}^{p-1} \frac{1}{m_i} e^2 \ .$$

Eventually, we get

$$|L(f_1) - L(f_2)| \le U e + \sum_{i=0}^{p-1} \frac{1}{m_i} e^2 \ . \tag{B.3}$$

□

Lemma 15. *Let f_1 and f_2 be two concave piecewise affine functions defined on $[c, d] \subset \mathbb{R}$ such that $f_1 \le f_2 \le f_1 + e$ for some $e > 0$. Then*

$$|L(f_1) - L(f_2)| \le U e \ . \tag{B.4}$$

where $U = \max(\varphi'(\alpha), \varphi'(\alpha) - 2\varphi'(\beta))$ with α, resp. β, lying between the slopes of the first, resp. last, segments of $\mathcal{C}(f_1)$ and $\mathcal{C}(f_2)$.

Proof. Let $\sigma = (x_k)_{k=0}^{p}$ be a common partition for f_1 and f_2. We write m_k for $x_{k+1} - x_k$ and $s_{1,k}$, resp. $s_{2,k}$, for the slope of f_1, resp. f_2, on the interval $[x_k, x_{k+1}]$. Since f_1 and f_2 are concave, the sequences $(s_{1,k})$ and $(s_{2,k})$ are monotonically non-increasing. Then,

$$L(f_1) - L(f_2) = \sum_{k=0}^{p-1} m_k (\varphi(s_{1,k}) - \varphi(s_{2,k})) = \sum_{k=0}^{p-1} \varphi'(z_k) m_k (s_{1,k} - s_{2,k}) \ ,$$

where $z_k \in (s_{1,k}, s_{2,k})$.

Let $i < j$ be two integers in $[\![0, p-1]\!]$. Since $s_{1,i} > s_{1,j}$, $s_{2,i} > s_{2,j}$ and, by definition, $\varphi'(z_i)$ and $\varphi'(z_j)$ are the slopes of two chords of the convex curve $\mathcal{C}(\varphi)$ between the points of abscissas $s_{1,i}$, $s_{2,i}$ for the former and between the points of abscissas $s_{1,j}$, $s_{2,j}$ for the latter, we derive that $\varphi'(z_i) > \varphi'(z_j)$. Thereby, the sequence $\big(\varphi'(z_k)\big)$ is monotonically non-increasing.

Now, from Lemma 13, taking

$$\begin{aligned} c_k &= m_k (s_{1,k} - s_{2,k}) \\ &= (f_1(x_{k+1}) - f_2(x_{k+1})) - (f_1(x_k) - f_2(x_k)), \\ u_k &= \varphi'(z_k) - \varphi'(z_{p-1}) \text{ and} \\ I &= [-e, e] \ , \end{aligned}$$

we derive from (12) that

$$\begin{aligned} |L(f_1) - L(f_2)| &= \left| \sum_{k=0}^{p-1} (u_k + \varphi'(z_{p-1})) c_k \right| \\ &\le \left| \sum_{k=0}^{p-1} u_k c_k \right| + |\varphi'(z_{p-1})| \sum_{k=0}^{p-1} c_k \\ &\le u_0 e + |\varphi'(z_{p-1})| \, e \\ &\le U e \ , \end{aligned}$$

where $U = \varphi'(z_0) - \varphi'(z_{p-1}) + |\varphi'(z_{p-1})| = \max(\varphi'(z_0), \varphi'(z_0) - 2\varphi'(z_{p-1}))$. $\square$

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